

# The Eigencurve is Proper

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## Abstract

We prove that the Coleman-Mazur eigencurve  $\mathcal{C}_{p,N}$  is proper over the weight space for any prime  $p$  and tame level  $N$ .

## 1 Introduction

Let  $p$  be a prime number. The purpose of this paper is to answer the following question raised by Coleman and Mazur in [CM98]:

Does there exist a  $p$ -adic family of finite slope overconvergent eigenforms, parameterized by a punctured disk, that converges to an overconvergent eigenform at the puncture which is of infinite slope?

In [*loc. cit.*], Coleman and Mazur constructed a  $\mathbb{Q}_p$ -rigid analytic curve over the weight space whose  $\mathbb{C}_p$ -points parametrize all finite slope overconvergent  $p$ -adic eigenforms of tame level 1. They call it the  *$p$ -adic eigencurve* of tame level 1. Such a construction was later generalized to all tame levels by Buzzard [Buz07]. In this framework, as suggested by Buzzard and Calegari [BuCa06], one can formulate the question of Coleman and Mazur as whether the projection from the eigencurve to the weight space satisfies the *valuative criterion for properness*.

In the past decade, some progresses have been made towards this problem. In [BuCa06], the properness was proved for  $p = 2$  and  $N = 1$ . In [Cal08], the properness was proved at integral weights in the center of  $\mathcal{W}_N$ . In this paper, we will show that for all primes  $p$  and tame levels  $N$ , the answer to this question is yes. More precisely, the main result of this paper is the following theorem.

**Theorem 1.1.** *Let  $\mathcal{C}_{p,N}$  be the Coleman-Mazur eigencurve of tame level  $N$ , and let  $\pi : \mathcal{C}_{p,N} \rightarrow \mathcal{W}_N$  denote the natural projection to the weight space. Let  $D$  be the closed unit disk over some finite extension  $L$  over  $\mathbb{Q}_p$ , and let  $D^*$  be the punctured disk with the origin removed. Suppose  $h : D^* \rightarrow \mathcal{C}_{p,N}$  is a morphism of rigid analytic spaces such that  $\pi \circ h$  extends to  $D$ .<sup>1</sup> Then  $h$  extends to a morphism  $\tilde{h} : D \rightarrow \mathcal{C}$  compatible with  $\pi \circ h$ .*

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<sup>1</sup>In fact, Lemma 2.1 implies that any rigid analytic map  $D^* \rightarrow \mathcal{W}_N$  extends uniquely to a rigid analytic map  $D \rightarrow \mathcal{W}_N$ ; hence this condition is always satisfied.

We have to point out that although this property is named “properness of the eigencurve”, the projection  $\pi$  is actually not proper in the sense of rigid analytic geometry because it is of infinite degree.

In the rest of the introduction we will sketch the steps to Theorem 1.1 and the structure of the paper. By [CM98], there exists a family of  $p$ -adic representations  $V_{\tilde{\mathcal{C}}}$  of  $G_{\mathbb{Q}}$  over the normalization  $\tilde{\mathcal{C}}_{p,N}$  of  $\mathcal{C}_{p,N}$  interpolating (the semi-simplifications of) the Galois representations associated to classical eigenforms. Granting this global input, our approach to Theorem 1.1 is purely local and Galois theoretical. Especially, we make extensive use of the recent advances [BeCo08], [KL10], [Liu12], [KPX] and [Bel13] on  $p$ -adic Hodge theory in arithmetic families. In §3, we will give a brief review on the family version of the functors  $D_{\text{dR}}^+$ ,  $D_{\text{crys}}^+$ ,  $D_{\text{rig}}^+$  and  $D_{\text{dif}}^+$  for the sake of the reader.

By its construction, one may regard the eigencurve  $\mathcal{C}_{p,N}$  as an analytic subspace of  $X_p \times \mathbb{G}_m$ , where  $X_p$  is the deformation space of the pseudo-representations associated to all (finitely many)  $p$ -modular, tame level  $N$ , residue representations of  $G_{\mathbb{Q}}$ . In §2, we will show that the composition  $D^* \rightarrow \mathcal{C}_{p,N} \hookrightarrow X_p \times \mathbb{G}_m \rightarrow X_p$  extends to a morphism  $D \rightarrow X_p$  of rigid analytic spaces. As a consequence, we obtain a family of pseudo-representations on  $D$  by pulling back the universal pseudo-representation over  $X_p$ . We then apply a result of [CM98] to convert it to a family of  $p$ -adic representations  $V_D$  over  $D$ .

The difficult part is to show that the composition  $D^* \rightarrow \mathcal{C}_{p,N} \hookrightarrow X_p \times \mathbb{G}_m \rightarrow \mathbb{G}_m$  extends to a morphism on  $D$ . This amounts to show that the pullback of the  $U_p$ -eigenvalues, which is denoted by  $\alpha$ , is nonzero at the puncture. We achieve this by showing that the specialization of  $V_D$  at the puncture has a nonzero crystalline period with Frobenius eigenvalue  $\alpha(0)$ . To this end, we compare the positive crystalline and de Rham periods of  $V_D^*$ . Recall that it was proved by Kisin that the positive crystalline and de Rham periods of  $V_{\tilde{\mathcal{C}}}^*$  coincides on “ $Y$ -small” affinoid subdomain of  $\tilde{\mathcal{C}}_{p,N}$  [Kis03]. This property was later strengthened to all affinoid subdomains by the work of the second author [Liu12]. In §4, using the results of [Liu12], we show that  $D_{\text{crys}}^+(V_R^*)^{\varphi=\alpha}$  is a locally free  $R$ -module of rank 1 and  $D_{\text{crys}}^+(V_R^*)^{\varphi=\alpha} = D_{\text{dR}}^+(V_R^*)$  for any affinoid subdomain  $M(R)$  of  $D^*$ , where  $V_R$  is the restriction of  $V_D$  on  $M(R)$ . A simple, but crucial, observation is that this property forces  $D_{\text{crys}}^+(V_D^*) = D_{\text{dR}}^+(V_D^*)$ ; we will show this in §5. In §6, we first apply the flat base change property of de Rham periods [Bel13] to show that  $D_{\text{dR}}^+(V_D^*)$  is nonzero. We then conclude by showing that the specialization of  $D_{\text{dR}}^+(V_D^*)$  at the puncture gives rise to the desired crystalline periods.

## Notations

Let  $p$  be a prime number. Choose a compatible system of primitive  $p$ -power roots of unity  $(\zeta_{p^n})_{n \geq 0}$ . Namely, each  $\zeta_{p^n}$  is a primitive  $p^n$ -th root of unity and  $\zeta_{p^{n+1}}^p = \zeta_{p^n}$ . Let  $\mathbb{Q}_p(\zeta_{p^\infty}) = \bigcup_{n \geq 1} \mathbb{Q}_p(\zeta_{p^n})$ . Let  $G_{\mathbb{Q}_p} = \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$  and  $\Gamma = \Gamma_{\mathbb{Q}_p} = \text{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p)$ . Let  $\Sigma$  be the finite set of places of  $\mathbb{Q}$  consisting of the infinite place and the places dividing  $pN$ , and let  $G_{\mathbb{Q},\Sigma}$  be the absolute Galois group of the maximal extension of  $\mathbb{Q}$  which is unramified outside the places of  $\Sigma$ .

For a topological group  $G$  and a rigid analytic space  $X$  over  $\mathbb{Q}_p$ , by a *family of  $p$ -adic representations of  $G$  on  $X$*  we mean a locally free coherent  $\mathcal{O}_X$ -module  $V_X$  equipped with a continuous  $\mathcal{O}_X$ -linear  $G$ -action, and we denote its dual by  $V_X^*$ . When  $X = M(S)$  is an affinoid space over  $\mathbb{Q}_p$ , we also call a family of  $p$ -adic representations of  $G$  on  $X$  an  *$S$ -linear  $G$ -representation*. If  $M(R) \subset M(S)$  is an affinoid subdomain and  $V_S$  is a family of representation on  $M(S)$ , we write  $V_R$  for the base change of  $V_S$  from  $S$  to  $R$ . Finally, for every  $x \in M(S)$ , we write  $V_x$  to denote the specialization  $V_S \otimes_S k(x)$  of  $V_S$  at  $x$ .

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## 2 The eigencurve $\mathcal{C}_{p,N}$

For a classical eigenform  $f$ , let  $\rho_f$  denote the  $p$ -adic Galois representation of  $G_{\mathbb{Q}}$  associated to  $f$ . Recall that a  $p$ -modular, tame level  $N$ , residual representation means a two dimensional  $G_{\mathbb{Q}}$ -representation  $\overline{V}$  over  $\overline{\mathbb{F}}_p$  which is isomorphic to the reduction of  $\rho_f$  for some eigenform  $f$  of level  $\Gamma_1(Np^m)$  with  $m \geq 1$ . The Eichler-Shimura relation implies that the  $G_{\mathbb{Q}}$ -action on  $\overline{V}$  factors through  $G_{\mathbb{Q},\Sigma}$ . Let  $R_{\overline{V}}$  be the universal deformation ring of the pseudo-representation associated to  $\overline{V}$ , and let  $r_{\overline{V}}$  be the associated rank 2 universal pseudo-representation. Let  $X_{\overline{V}}$  be the rigid analytic space over  $\mathbb{Q}_p$  associated to  $R_{\overline{V}}$ . For an eigenform  $f$  of level  $\Gamma_1(Np^m)$ , if the reduction of  $\rho_f$  is isomorphic to  $\overline{V}$ , and if the  $U_p$ -eigenvalue  $\alpha_f$  of  $f$  is nonzero, it gives rise to a modular point  $(\rho_f, \alpha_f^{-1})$  in  $X_{\overline{V}} \times \mathbb{G}_m$ . Let  $X_p = \coprod X_{\overline{V}}$  where  $\overline{V}$  runs through all (finitely many)  $p$ -modular tame level  $N$  residual representations. By the work of Coleman and Mazur [CM98], one may regard the tame level  $N$  eigencurve  $\mathcal{C}_{p,N}$  as an analytic subspace of  $X_p \times \mathbb{G}_m$ , whose reduction is equal to the Zariski closure of all such modular points  $(\rho_f, \alpha_f^{-1})$ . The  $\mathbb{C}_p$ -points of  $\mathcal{C}_{p,N}$  correspond bijectively to finite slope overconvergent  $p$ -adic eigenforms of tame level  $N$ .

We obtain a rank 2 pseudo-representation of  $G_{\mathbb{Q},\Sigma}$  on the eigencurve  $\mathcal{C}_{p,N}$  by pulling back the universal pseudo-representations on  $X_p$ . Let  $\tilde{\mathcal{C}}_{p,N}$  be the normalization of  $\mathcal{C}_{p,N}$ . By [CM98, Theorem 5.1.2] (see the remark after it), any rank 2 pseudo-representations of  $G_{\mathbb{Q},\Sigma}$  on a smooth rigid analytic curve over  $\mathbb{Q}_p$  can be converted naturally to a family of  $p$ -adic representations of  $G_{\mathbb{Q},\Sigma}$ . Thus there exists a family of  $G_{\mathbb{Q},\Sigma}$ -representations  $V_{\tilde{\mathcal{C}}}$  of rank 2 on  $\tilde{\mathcal{C}}_{p,N}$  whose associated pseudo-representation is isomorphic to the pullback of the pseudo-representation on  $\mathcal{C}_{p,N}$ . We may assume that the given map  $h$  is dominant, otherwise the situation would become trivial. Since  $D^*$  is smooth, it follows that the given map  $h : D^* \rightarrow \mathcal{C}_{p,N}$  factors through  $\tilde{\mathcal{C}}_{p,N}$ . By abuse of notation we still denote the resulting map  $D^* \rightarrow \tilde{\mathcal{C}}_{p,N}$  by  $h$ . Let  $V_{D^*}$  denote the pullback of  $V_{\tilde{\mathcal{C}}}$  along  $h$ . The goal of this section is to show that the composition  $u : D^* \rightarrow \mathcal{C}_{p,N} \hookrightarrow X_p \times \mathbb{G}_m \rightarrow X_p$  extends to a morphism on the entire disk  $D$ . Before proceeding, we first make the following observation.

**Lemma 2.1.** *Let  $F \in \mathcal{O}(D^*)$ . If  $|F(x)|$  is bounded for all  $x \in D^*$ ,  $F$  extends uniquely to an element of  $\mathcal{O}(D)$ .*

*Proof.* The uniqueness is obvious. After scaling, we may suppose  $|F(x)| \leq 1$  for any  $x \in D^*$ . Let  $D_{0,n} = M(\mathbb{Q}_p\langle T, p^n T^{-1} \rangle)$  be the closed annulus with outside radius 1 and inside radius  $p^{-n}$ . Then  $|F| \leq 1$  on  $D_{0,n}$  for all  $n \geq 1$ . This implies  $F \in \mathbb{Z}_p\langle T, p^n T^{-1} \rangle$  for all  $n \geq 1$ . Hence

$$F \in \bigcap_{n \geq 1} \mathbb{Z}_p\langle T, p^n T^{-1} \rangle = \mathbb{Z}_p\langle T \rangle,$$

yielding  $F \in \mathcal{O}(D)$ . □

**Proposition 2.2.** *The morphism  $u$  extends to a morphism of rigid analytic spaces  $\tilde{u} : D \rightarrow X_p$ .*

*Proof.* Since  $D^*$  is connected, it maps to  $X_{\overline{V}}$  for some  $\overline{V}$ . Recall that  $X_{\overline{V}}$  is the generic fibre of  $\mathrm{Spf}(R_{\overline{V}})$ . Thus for any  $x \in D^*$  and  $t \in R_{\overline{V}}$ , we have  $|u^*(t)(x)| = |t(u(x))| \leq 1$ . By Lemma 2.1,

$u^*(t)$  extends to an element of  $\mathbb{Z}_p\langle T \rangle$ . We therefore obtain a continuous morphism  $R_{\overline{V}} \rightarrow \mathbb{Z}_p\langle T \rangle$ . This yields the desired extension.  $\square$

We denote by  $r_{D^*}$  the pseudo-representation associated to  $V_{D^*}$ . By the construction of  $V_{D^*}$ , we see that  $r_{D^*}$  is isomorphic to the pullback of the universal pseudo-representation on  $X_p$  along  $u$ .

**Corollary 2.3.** *The family of  $p$ -adic representations  $V_{D^*}$  extends to  $D$ .*

*Proof.* Pulling back the universal pseudo-representation on  $X_p$  along  $\tilde{u}$ , which is given by Proposition 2.2, we obtain a pseudo-representation  $r_D$  of  $G_{\mathbb{Q}, \Sigma}$  on  $D$  which extends  $r_{D^*}$ . Since  $D$  is a smooth rigid analytic curve over  $\mathbb{Q}_p$ , as explained earlier, by [CM98, Theorem 5.1.2], one can convert  $r_D$  to a family of  $p$ -adic representations on  $D$ , yielding the desired extension.  $\square$

Henceforth we denote by  $V_D$  the extended family of  $p$ -adic representations of  $G_{\mathbb{Q}, \Sigma}$  on  $D$  given by Corollary 2.3.

### 3 Families of $p$ -adic representations

In this section, we give a brief review on various  $p$ -adic Hodge theoretic functors for families of  $p$ -adic representations of  $G_{\mathbb{Q}_p}$ . We refer the reader to [BeCo08] and [KL10] for more details.

#### 3.1 The modules $D_{\text{dR}}^+(V_S)$ and $D_{\text{crys}}^+(V_S)$

Let  $S$  be a  $\mathbb{Q}_p$ -affinoid algebra, and let  $V_S$  be an  $S$ -linear  $G_{\mathbb{Q}_p}$ -representation.

Let  $\mathbf{B}_{\text{dR}}^+$  and  $\mathbf{B}_{\text{crys}}^+$  be the de Rham and crystalline period rings used in  $p$ -adic Hodge theory. For each  $k > 0$ ,  $\mathbf{B}_{\text{dR}}^+/(t^k)$  is naturally a  $\mathbb{Q}_p$ -Banach space. This gives a Fréchet topology on

$$\mathbf{B}_{\text{dR}}^+ = \varprojlim_k \mathbf{B}_{\text{dR}}^+/(t^k).$$

So we can define  $S \hat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\text{dR}}^+ = \varprojlim_k S \hat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\text{dR}}^+/(t^k)$ . We can also define  $S \hat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\text{crys}}^+$  as  $\mathbf{B}_{\text{crys}}^+$  has a natural  $\mathbb{Q}_p$ -Banach space structure. For an  $S$ -linear representation  $V_S$  of  $G_{\mathbb{Q}_p}$ , following [BeCo08], we set

$$D_{\text{dR}}^+(V_S) = ((S \hat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\text{dR}}^+) \otimes_S V_S)^{G_{\mathbb{Q}_p}},$$

and

$$D_{\text{crys}}^+(V_S) = ((S \hat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\text{crys}}^+) \otimes_S V_S)^{G_{\mathbb{Q}_p}}.$$

The following proposition, which is due to Bellovin, ensures the flat base change property of the functor  $D_{\text{dR}}^+$ .

**Proposition 3.1.** *Let  $V_S$  be an  $S$ -linear  $G_{\mathbb{Q}_p}$ -representation. If  $f : S \rightarrow S'$  is a flat morphism of  $\mathbb{Q}_p$ -affinoid algebras, then*

$$D_{\text{dR}}^+(V_S) \otimes_S S' \xrightarrow{\sim} D_{\text{dR}}^+(V \otimes_S S').$$

*Proof.* See the proof of [Bel13, Proposition 4.3.7].  $\square$

### 3.2 The modules $D_{\text{rig}}^{\dagger}(V_S)$ and $D_{\text{dif}}^+(V_S)$

Let  $\mathbf{B}_{\mathbb{Q}_p}^{\dagger,s}, \mathbf{B}_{\mathbb{Q}_p}^{\dagger}, \mathbf{B}_{\text{rig},\mathbb{Q}_p}^{\dagger,s}, \mathbf{B}_{\text{rig},\mathbb{Q}_p}^{\dagger}$  be the period rings in the theory of  $(\varphi, \Gamma)$ -modules introduced in [Ber02]. In [BeCo08], Berger and Colmez construct the family version of overconvergent  $(\varphi, \Gamma)$ -modules functor for free  $S$ -linear representations. This functor is later generalized to general  $S$ -linear representations by Kedlaya and the second author in [KL10].

More precisely, let  $V_S$  be an  $S$ -linear  $G_{\mathbb{Q}_p}$ -representation of rank  $d$ . For sufficiently large  $s$ , one can construct a locally free  $S\widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\mathbb{Q}_p}^{\dagger,s}$ -module  $D^{\dagger,s}(V_S)$  of rank  $d$  such that for any  $x \in M(S)$ ,  $D^{\dagger,s}(V_S) \otimes_S S/\mathfrak{m}_x$  is naturally isomorphic to  $D^{\dagger,s}(V_x)$ . We set

$$(S\widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\mathbb{Q}_p}^{\dagger}) = \bigcup_s S\widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\mathbb{Q}_p}^{\dagger,s}$$

and define

$$D^{\dagger}(V_S) = (S\widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\mathbb{Q}_p}^{\dagger}) \otimes_{S\widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\mathbb{Q}_p}^{\dagger,s}} D^{\dagger,s}(V_S) = \bigcup_s D^{\dagger,s}(V_S).$$

This is a locally free  $S\widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\mathbb{Q}_p}^{\dagger}$ -module of rank  $d$  and specializes to  $D^{\dagger}(V_x)$  for any  $x \in M(S)$ . Moreover,  $D^{\dagger}(V_S)$  is equipped with commuting semilinear  $\varphi, \Gamma$ -actions. This makes  $D^{\dagger}(V_S)$  an *étale*  $(\varphi, \Gamma)$ -module over  $S\widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\mathbb{Q}_p}^{\dagger}$  in the sense of [KL10, Definition 2.8].

For sufficiently large  $s$ , we define

$$D_{\text{rig}}^{\dagger,s}(V_S) = (S\widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\text{rig},\mathbb{Q}_p}^{\dagger,s}) \otimes_{S\widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\text{rig},\mathbb{Q}_p}^{\dagger,s}} D^{\dagger,s}(V_S).$$

We set

$$S\widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\text{rig},\mathbb{Q}_p}^{\dagger} = \bigcup_s S\widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\text{rig},\mathbb{Q}_p}^{\dagger,s},$$

and define

$$D_{\text{rig}}^{\dagger}(V_S) = (S\widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\text{rig},\mathbb{Q}_p}^{\dagger}) \otimes_{S\widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\text{rig},\mathbb{Q}_p}^{\dagger,s}} D_{\text{rig}}^{\dagger,s}(V_S) = \bigcup_s D_{\text{rig}}^{\dagger,s}(V_S).$$

Then  $D_{\text{rig}}^{\dagger}(V_S)$  is an *étale family* of  $(\varphi, \Gamma)$ -module over  $S\widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\text{rig},\mathbb{Q}_p}^{\dagger}$  in the sense of [KL10, Definition 6.3].

Recall that for  $0 < s \leq r_n = p^{n-1}(p-1)$ , one has the localization map

$$\iota_n : \mathbf{B}_{\text{rig},\mathbb{Q}_p}^{\dagger,s} \rightarrow \mathbb{Q}_p(\zeta_{p^n})[[t]].$$

This induces a continuous map  $S\widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\text{rig},\mathbb{Q}_p}^{\dagger,s} \rightarrow S\widehat{\otimes}_{\mathbb{Q}_p} \mathbb{Q}_p(\zeta_{p^n})[[t]]$ . Define

$$D_{\text{dif}}^{+,n}(V_S) = (S\widehat{\otimes}_{\mathbb{Q}_p} \mathbb{Q}_p(\zeta_{p^n})[[t]]) \otimes_{\iota_n, S\widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\text{rig},\mathbb{Q}_p}^{\dagger,s}} D_{\text{rig}}^{\dagger,s}(V_S).$$

It is clear that  $D_{\text{dif}}^{+,n}(V_S)$  is a locally free  $S\widehat{\otimes}_{\mathbb{Q}_p} \mathbb{Q}_p(\zeta_{p^n})[[t]]$ -module of rank  $d$  equipped with a semilinear  $\Gamma$ -action.

Abusing the notation, we still denote by  $\iota_n$  the natural map  $\iota_n : D_{\text{rig}}^{\dagger,s}(V_S) \rightarrow D_{\text{dif}}^{+,n}(V_S)$ . We define

$$D_{\text{dif}}^+(V_S) = \bigcup_n D_{\text{dif}}^{+,n}(V_S).$$

The following theorem generalizes Berger's comparisons between  $p$ -adic Hodge theory and  $(\varphi, \Gamma)$ -modules functors to  $S$ -linear representations.

**Theorem 3.2.** [Bel13, Theorem 4.2.7, Theorem 4.2.8] For an  $S$ -linear representation  $V_S$  of  $G_{\mathbb{Q}_p}$ , we have  $D_{\text{dR}}^+(V_S) = D_{\text{dif}}^+(V_S)^\Gamma$  and  $D_{\text{crys}}^+(V_S) = (D_{\text{rig}}^+(V_S))^\Gamma$ .

## 4 Finite slope subspace of $D^*$

Let  $X$  be a reduced and separated rigid analytic space over  $\mathbb{Q}_p$ , and let  $V_X$  be a family of  $p$ -adic representations over  $X$  of  $G_{\mathbb{Q}_p}$  having 0 as a Hodge-Tate-Sen weight. We may write the Sen polynomial of  $V_X$  as  $uQ(u)$  where  $Q(u) \in \mathcal{O}(X)[u]$ . Let  $\alpha \in \mathcal{O}(X)^\times$ . Recall that in [Liu12], the second author introduces the notion of finite slope subspaces of  $X$  with respect to the pair  $(\alpha, V_X)$ , which refines the original definition of finite slope subspaces introduced by Kisin [Kis03].

**Definition 4.1.** For such a triple  $(X, \alpha, V_X)$ , we call an analytic subspace  $X_{fs} \subset X$  a *finite slope subspace* of  $X$  with respect to the pair  $(\alpha, V_X)$  if it satisfies the following conditions.

- (1) For every integer  $j \leq 0$ , the subspace  $(X_{fs})_{Q(j)}$  is scheme-theoretically dense in  $X_{fs}$ .
- (2) For any affinoid algebra  $R$  over  $\mathbb{Q}_p$  and morphism  $g : M(R) \rightarrow X$  which factors through  $X_{Q(j)}$  for every integer  $j \leq 0$ , the morphism  $g$  factors through  $X_{fs}$  if and only if the natural map

$$\iota_n : (D_{\text{rig}}^+(V_R))^{\varphi^f = g^*(\alpha), \Gamma=1} \rightarrow D_{\text{dif}}^{+,n}(V_R)^\Gamma \quad (4.1.1)$$

is an isomorphism for all  $n$  sufficiently large.

Furthermore, [Liu12, Theorem 3.3.1] ensures that  $X$  has a unique finite slope subspace  $X_{fs}$  associated to the pair  $(V_X, \alpha)$ .

Let  $\alpha_{\tilde{\mathcal{C}}} \in \mathcal{O}(\tilde{\mathcal{C}}_{p,N})^\times$  be the function of  $U_p$ -eigenvalues. Let  $\alpha_{\tilde{\mathcal{C}}} \in \mathcal{O}(\tilde{\mathcal{C}}_{p,N})^\times$  and  $\alpha \in \mathcal{O}(D^*)^\times$  be the pullbacks of  $\alpha_{\tilde{\mathcal{C}}}$ . Since the family of  $p$ -adic representations  $V_{\tilde{\mathcal{C}}}$  has 0 as a Hodge-Tate-Sen weight, we may write the Sen polynomial of  $V_{\tilde{\mathcal{C}}}^*$  as  $T(T - \kappa_{\tilde{\mathcal{C}}})$  for some  $\kappa_{\tilde{\mathcal{C}}} \in \mathcal{O}(\tilde{\mathcal{C}}_{p,N})$ . Let  $\kappa = h^*(\kappa_{\tilde{\mathcal{C}}}) \in (D^*)$ . It follows that the Sen polynomial of  $V_{D^*}^*$  is  $T(T - \kappa)$ .

**Proposition 4.2.** The finite slope subspace  $(D^*)_{fs}$  of the punctured disk  $D^*$  associated to  $(V_{D^*}^*, \alpha)$  is  $D^*$  itself.

*Proof.* To show the proposition, we just need to check that the triple  $(D^*, V_{D^*}^*, \alpha)$  satisfies the conditions (1) and (2) of Definition 4.1. Since the finite slope subspace associated to the pair  $(V_{\tilde{\mathcal{C}}}^*, \alpha_{\tilde{\mathcal{C}}})$  is  $\tilde{\mathcal{C}}$  itself by the main results of [Liu12], we know that the triple  $(\tilde{\mathcal{C}}, \alpha_{\tilde{\mathcal{C}}}, V_{\tilde{\mathcal{C}}}^*)$  satisfies the conditions (1) and (2). Hence  $\tilde{\mathcal{C}}_{(\kappa_{\tilde{\mathcal{C}}}-j)}$  is scheme-theoretically dense in  $\tilde{\mathcal{C}}$  for every  $j \leq 0$ . Since  $f$  is dominant and  $D^*$  is of dimension 1, we deduce that  $D_{(\kappa-j)}^* = f^{-1}(\tilde{\mathcal{C}}_{(\kappa_{\tilde{\mathcal{C}}}-j)})$  is scheme-theoretically dense in  $D^*$ . Thus the triple  $(D^*, V_{D^*}^*, \alpha)$  satisfies the condition (1). It follows immediately that the triple  $(D^*, V_{D^*}^*, \alpha)$  also satisfies the condition (2) because  $D_{(\kappa-j)}^*$  maps to  $\tilde{\mathcal{C}}_{(\kappa_{\tilde{\mathcal{C}}}-j)}$  for every  $j \leq 0$ .  $\square$

**Proposition 4.3.** For any affinoid subdomain  $M(R)$  of  $D^*$  and  $k > \log_p |\alpha^{-1}|_{\text{sp}}$ , where  $|\cdot|_{\text{sp}}$  denotes the spectral norm taken on  $M(R)$ , the natural map

$$(D_{\text{rig}}^+(V_R^*))^{\varphi=\alpha, \Gamma=1} \rightarrow (D_{\text{dif}}^+(V_R^*)/(t^k))^\Gamma$$

is an isomorphism. Furthermore,  $(D_{\text{rig}}^+(V_R^*))^{\varphi=\alpha, \Gamma=1}$  is a locally free  $R$ -module of rank 1.

*Proof.* Since the finite slope subspace of  $D^*$  is itself, it follows immediately from [Liu12, Theorem 3.3.4] that the given map is an isomorphism. Note that  $M(R)$  is smooth of dimension 1. We deduce that the  $R$ -module  $(D_{\text{dif}}^+(V_R)/(t^k))^\Gamma$  is locally free as it is finite and torsion free. Moreover, for any  $x \in M(R)$  with non-integral weight, by [Liu12, Corollary 1.5.6], the natural map

$$(D_{\text{dif}}^+(V_R^*)/(t^k))^\Gamma \otimes_R k(x) \rightarrow (D_{\text{dif}}^+(V_x^*)/(t^k))^\Gamma$$

is an isomorphism. It is clear that the right hand side is of  $k(x)$ -dimension 1. Since the subset of points with non-integral weights is Zariski dense in  $M(R)$ , we conclude that  $(D_{\text{dif}}^+(V_R^*)/(t^k))^\Gamma$  is a locally free  $R$ -module of rank 1, and so is  $(D_{\text{rig}}^\dagger(V_R^*))^{\varphi=\alpha, \Gamma=1}$ .  $\square$

**Corollary 4.4.** *For any affinoid subdomain  $M(R)$  of  $D^*$ , the natural map*

$$D_{\text{crys}}^+(V_R^*)^{\varphi=\alpha} \rightarrow D_{\text{dR}}^+(V_R^*)$$

*is an isomorphism. Furthermore, they are locally free  $R$ -modules of rank 1.*

*Proof.* The previous proposition implies that for sufficiently large  $k$ , the natural map

$$(D_{\text{rig}}^\dagger(V_R^*))^{\varphi=\alpha, \Gamma=1} \rightarrow (D_{\text{dif}}^+(V_R^*)/(t^k))^\Gamma$$

is an isomorphism, yielding that the natural map

$$(D_{\text{rig}}^\dagger(V_R^*))^{\varphi=\alpha, \Gamma=1} \rightarrow D_{\text{dif}}^+(V_R^*)^\Gamma = \varprojlim_k (D_{\text{dif}}^+(V_R^*)/(t^k))^\Gamma$$

is an isomorphism. We conclude by applying Theorem 3.2.  $\square$

## 5 De Rham periods vs crystalline periods

The goal of this section is to show  $D_{\text{dR}}^+(V_D^*) = D_{\text{crys}}^+(V_D^*)$ . We first fix some notations which will be used in the rest of the paper. Let  $S = \mathbb{Q}_p\langle T \rangle$ . For any  $n \geq 0$  (resp.  $n' > n \geq 0$ ), let  $S_n = \mathbb{Q}_p\langle p^{-n}T \rangle$  (resp.  $S_{n,n'} = \mathbb{Q}_p\langle p^{-n}T, p^{-n'}T^{-1} \rangle$ ). Let  $V_n$  (resp.  $V_{n,n'}$ ) be the restriction of  $V_D$  on  $M(S_n)$  (resp.  $M(S_{n,n'})$ ).

**Definition 5.1.** Let  $A$  be a Banach algebra over  $\mathbb{Q}_p$ .

- (i) For any  $n \geq 0$ , define the Banach algebra  $A\langle p^{-n}T \rangle$  to be the ring of formal power series  $\sum_{i \in \mathbb{N}} a_i T^i$  with  $a_i \in A$  and such that  $|a_i|p^{-ni} \rightarrow 0$  as  $i \rightarrow \infty$ . It is equipped with a Banach norm  $|\sum_{i \in \mathbb{N}} a_i T^i| = \sup |a_i|p^{-ni}$ .
- (ii) For any  $n' > n \geq 0$ , define the Banach algebra  $A\langle p^{-n}T, p^{-n'}T^{-1} \rangle$  to be the ring of Laurent series  $\sum_{i \in \mathbb{Z}} a_i T^i$  with  $a_i \in A$  and such that  $|a_i|p^{-ni} \rightarrow 0$  as  $i \rightarrow \infty$  and  $|a_i|p^{-n'i} \rightarrow 0$  as  $i \rightarrow -\infty$ . It is equipped with a Banach norm  $|\sum_{i \in \mathbb{Z}} a_i T^i| = \max\{\sup |a_i|p^{-ni}, \sup |a_i|p^{-n'i}\}$ .

Using the facts that  $\{p^{-ni}T^i\}_{i \in \mathbb{N}}$  forms an orthonormal basis of  $S_n$  and  $\{p^{-ni}T^i, p^{-n'(i+1)}T^{-i-1}\}_{i \in \mathbb{N}}$  forms an orthonormal basis of  $S_{n,n'}$ , we deduce the following lemma.

**Lemma 5.2.** *Let  $A$  be a  $\mathbb{Q}_p$ -Banach algebra. For any  $n \geq 0$ , we have natural identification of Banach algebras*

$$\eta_{n,A} : S_n \widehat{\otimes}_{\mathbb{Q}_p} A \xrightarrow{\sim} A\langle p^{-n}T \rangle.$$

*Similarly, for any  $n' > n \geq 0$ , we have natural identification of Banach algebras*

$$\eta_{n,n',A} : S_{n,n'} \widehat{\otimes}_{\mathbb{Q}_p} A \xrightarrow{\sim} A\langle p^{-n}T, p^{n'}T^{-1} \rangle.$$

**Definition 5.3.** Let  $A = \varprojlim_{j \in J} A_j$  be a Fréchet algebra where  $A_j$ 's are  $\mathbb{Q}_p$ -Banach algebras.

- (i) Define the Fréchet algebra  $A\langle p^{-n}T \rangle$  to be the inverse limit of Banach algebras  $A_j\langle p^{-n}T \rangle$ .
- (ii) For any  $n' > n \geq 0$ , define the Fréchet algebra  $A\langle p^{-n}T, p^{n'}T^{-1} \rangle$  to be the inverse limit of Banach algebras  $A_j\langle p^{-n}T, p^{n'}T^{-1} \rangle$ .

Note that the natural inclusions  $A_j\langle p^{-n}T \rangle \hookrightarrow A_j[[T]]$  induces an injective map

$$A\langle p^{-n}T \rangle = \varprojlim_{j \in J} A_j\langle p^{-n}T \rangle \hookrightarrow \varprojlim_{j \in J} A_j[[T]] = A[[T]].$$

Thus one may naturally identify  $A\langle p^{-n}T \rangle$  as a subring of  $A[[T]]$ . Similarly, one can naturally identify  $A\langle p^{-n}T, p^{n'}T^{-1} \rangle$  as a subset of  $A[[T, T^{-1}]]$ , which is the set of Laurent series with coefficients in  $A$ ; note that  $A[[T, T^{-1}]]$  is not a ring!

**Lemma 5.4.** *Keep notations as in Definition 5.3. For any  $n \geq 0$ , we have natural identification of Fréchet algebras*

$$\eta_{n,A} : S_n \widehat{\otimes}_{\mathbb{Q}_p} A \xrightarrow{\sim} A\langle p^{-n}T \rangle.$$

*Similarly, for any  $n' > n \geq 0$ , we have natural identification of Fréchet algebras*

$$\eta_{n,n',A} : S_{n,n'} \widehat{\otimes}_{\mathbb{Q}_p} A \xrightarrow{\sim} A\langle p^{-n}T, p^{n'}T^{-1} \rangle.$$

*Proof.* Apply the previous lemma to the  $\mathbb{Q}_p$ -Banach algebras  $A_j$  and take inverse limits. □

In particular, Lemma 5.2 applies to  $A = \mathbf{B}_{\text{crys}}^+$  and Lemma 5.4 applies to  $A = \mathbf{B}_{\text{dR}}^+ = \varprojlim \mathbf{B}_{\text{dR}}^+(t^i)$ .

**Lemma 5.5.** (i) *For any  $n \geq 0$ , the continuous map  $\mathbf{B}_{\text{crys}}^+ \rightarrow \mathbf{B}_{\text{dR}}^+$  induces a natural inclusion  $\mathbf{B}_{\text{crys}}^+\langle p^{-n}T \rangle \hookrightarrow \mathbf{B}_{\text{dR}}^+\langle p^{-n}T \rangle$ .*

(ii) *For any  $n' > n \geq 0$ , the continuous map  $\mathbf{B}_{\text{crys}}^+ \rightarrow \mathbf{B}_{\text{dR}}^+(t^i)$  induces a natural inclusion  $\mathbf{B}_{\text{crys}}^+\langle p^{-n}T, p^{n'}T^{-1} \rangle \hookrightarrow \mathbf{B}_{\text{dR}}^+\langle p^{-n}T, p^{n'}T^{-1} \rangle$ .*

*Proof.* By the commutative diagram

$$\begin{array}{ccc} \mathbf{B}_{\text{crys}}^+\langle p^{-n}T \rangle & \xrightarrow{\quad} & \mathbf{B}_{\text{dR}}^+\langle p^{-n}T \rangle \\ \downarrow & & \downarrow \\ \mathbf{B}_{\text{crys}}^+[[T]] & \xrightarrow{\quad} & \mathbf{B}_{\text{dR}}^+[[T]], \end{array}$$



we see that the composition  $\mathbf{B}_{\text{crys}}^+ \langle p^{-n}T \rangle \rightarrow \mathbf{B}_{\text{dR}}^+ \langle p^{-n}T \rangle \rightarrow \mathbf{B}_{\text{dR}}^+[[T]]$  is injective. Hence the natural map  $\mathbf{B}_{\text{crys}}^+ \langle p^{-n}T \rangle \rightarrow \mathbf{B}_{\text{dR}}^+ \langle p^{-n}T \rangle$  is injective. The proof of (ii) is similar.  $\square$

As a consequence of Lemma 5.5, we may naturally identify  $S_n \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\text{crys}}^+$  (resp.  $S_{n,n'} \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\text{crys}}^+$ ) as a subring of  $S_n \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\text{dR}}^+$  (resp.  $S_{n,n'} \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\text{dR}}^+$ ).

**Lemma 5.6.** *For any  $x \in S_n \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\text{dR}}^+$ , if its image in  $S_{n,n'} \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\text{dR}}^+$  belongs to  $S_{n,n'} \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\text{crys}}^+$ , then  $x \in S_n \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\text{crys}}^+$ .*

*Proof.* By the previous lemmas, we may regard all the rings involved as subsets of  $\mathbf{B}_{\text{dR}}^+[[T, T^{-1}]]$ . It follows from the assumption that

$$x \in \mathbf{B}_{\text{dR}}^+ \langle p^{-n}T \rangle \cap \mathbf{B}_{\text{crys}}^+ \langle p^{-n}T, p^{n'}T^{-1} \rangle \subseteq \mathbf{B}_{\text{dR}}^+[[T]] \cap \mathbf{B}_{\text{crys}}^+ \langle p^{-n}T, p^{n'}T^{-1} \rangle = \mathbf{B}_{\text{crys}}^+ \langle p^{-n}T \rangle,$$

yielding the desired result.  $\square$

**Corollary 5.7.**  $D_{\text{crys}}^+(V_D^*) = D_{\text{dR}}^+(V_D^*)$ .

*Proof.* By Corollary 4.4, we get  $D_{\text{crys}}^+(V_{0,1}^*) = D_{\text{dR}}^+(V_{0,1}^*)$ . Applying the previous lemma, we deduce that  $D_{\text{dR}}^+(V_D^*) \subseteq D_{\text{crys}}^+(V_D^*)$ ; hence  $D_{\text{dR}}^+(V_D^*) = D_{\text{crys}}^+(V_D^*)$ .  $\square$

## 6 Proof of Theorem 1.1

Applying Proposition 3.1, we first obtain  $D_{\text{dR}}^+(V_D^*) \otimes_S S_{0,1} \xrightarrow{\sim} D_{\text{dR}}^+(V_{0,1}^*)$ . By Corollary 4.4, we see that  $D_{\text{dR}}^+(V_{0,1}^*)$  is a locally free  $S_{0,1}$ -module of rank 1. In particular, this implies  $D_{\text{dR}}^+(V_D^*) \neq 0$ . Now pick a nonzero element  $e \in D_{\text{dR}}^+(V_D^*)$ . By dividing a suitable power of  $T$ , we may assume that the specialization  $e_0$  of  $e$  at the puncture 0 is nonzero. Note that  $e \in D_{\text{crys}}^+(V_D^*)$  by Corollary 5.7. Moreover, the image of  $e$  in  $D_{\text{crys}}^+(V_{0,1}^*)$  belongs to  $D_{\text{crys}}^+(V_{0,1}^*)^{\varphi=\alpha}$  by Corollary 4.4. That is,  $\varphi(e) = \alpha e$  on  $M(S_{0,1})$ . Note that the norms of the  $U_p$ -eigenvalues of overconvergent  $p$ -adic eigenforms are always less than or equal to 1. We thus conclude  $\alpha \in \mathcal{O}(D)$  by Lemma 2.1. Since  $M(S_{0,1})$  is Zariski dense in  $D$ , we must have  $\varphi(e) = \alpha e$  on the entire disk. In particular,  $\varphi(e_0) = \alpha(0)e_0$ . Since  $\varphi$  is injective on  $D_{\text{crys}}^+(V_0^*)$ , we conclude that  $\alpha(0) \neq 0$ . Thus  $\alpha \in \mathcal{O}(D)^\times$ .

Now we construct a map  $\tilde{h} : D \rightarrow X_p \times \mathbb{G}_m$  of rigid analytic spaces by sending  $x$  to  $(\tilde{u}(x), \alpha(x)^{-1})$  where  $\tilde{u}$  is given by Proposition 2.2. It is clear that  $\tilde{h}|_{D^*} = h$ . Since  $\mathcal{C}_{p,N}$  is an analytic subspace of  $X_p \times \mathbb{G}_m$ ,  $\tilde{h}^{-1}(\mathcal{C}_{p,N})$  is an analytic subspace of  $D$  containing  $D^*$ . This forces  $\tilde{h}^{-1}(\mathcal{C}_{p,N}) = D$ , confirming that  $\tilde{h}$  is the desired extension of  $h$ .

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